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Cramer Rule and Adjoint Method for Reduced Biquaternionic Linear Equations

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Abstract

In this paper, we introduce basic properties of the reduced biquaternion which can be regarded as generalization of complex numbers. By means of a complex representation of a reduced biquaternion matrix, we discuss in detail a Cramer rule and adjoint method for the reduced biquaternionic linear equations.

Keywords: reduced biquaternion, complex representation

1 Introduction

The reduced biquaternion can be seen as one of generalization of the complex numbers (see, for example, [1, 2]). It has been extensively applied in digital signal and image processing [4, 6]. The reduced biquaternion was introduced by Schtte and Wenzel [5], independently. They used the reduced biquaternion to the implementation of digital filter, and demonstrated that fourth order real filter can be realized by means of a first order reduced biquaternion filters.

In [3], the author proposed a Cramer rule for solving quaternionic linear equations using a complex representation of a quaternion matrix. In the present paper we first investigate in detail the properties of the reduced biquaternion in context of complex number representation. We then establish important properties of matrix representation of the reduced biquaternion which are corresponding properties of the complex matrices. By means of a complex representation of a reduced biquaternion matrix [2], we study in detail a Cramer rule and adjoint method for the reduced

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biquaternionic linear equations, which are different from the Cramer rule adjoint method for the quaternionic linear equations.

2 Reduced Biquaternion

Let $\mathbb{R}$ denotes the sets of real numbers and RB the reduced biquaternion over $\mathbb{R}$. Hence, every $q \in RB$ can be defined by

$$q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3;\ q_0, q_1, q_2, q_3 \in \mathbb{R}, \tag{1}$$

where basis elements $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ obey the following multiplication rules:

$$ij = ji = k, jk = kj = i, ki = ik = -j, i^2 = k^2 = -1, j^2 = 1. \tag{2}$$

According to equation (2) it is easily seen that the reduced biquaternion multiplication is commutative. Therefore, the multiplication of any reduced biquaternion

$$p = p_0 + ip_1 + jp_2 + kp_3$$

and

$$q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3,$$

is obtained by

$$\begin{aligned}
pq &= qp \\
&= p_0q_0 + ip_0q_1 + jp_0q_2 + kp_0q_3 + ip_1q_0 + i^2p_1q_1 + ijp_1q_2 + ikp_1q_3 \\
&\quad + jp_2q_0 + jip_2q_1 + j^2p_2q_2 + jkp_2q_3 + kp_3q_0 + kip_3q_1 + kjp_3q_2 + k^2p_3q_3 \\
&= p_0q_0 + ip_0q_1 + jp_0q_2 + kp_0q_3 + ip_1q_0 - p_1q_1 + kp_1q_2 - jp_1q_3 + jp_2q_0 \\
&= p_0q_0 + ip_0q_1 + jp_0q_2 + kp_0q_3 + ip_1q_0 - p_1q_1 + kp_1q_2 - jp_1q_3 + jp_2q_0 \\
&\quad + kp_2q_1 + p_2q_2 + ip_2q_3 + kp_3q_0 - jp_3q_1 + ip_3q_2 - p_3q_3 \\
&= (p_0q_0 - p_1q_1 + p_2q_2 - p_3q_3) + \boldsymbol{i}(p_0q_1 + p_1q_0 + p_2q_3 + p_3q_2) \\
&\quad + \boldsymbol{j}(p_0q_2 - p_1q_3 + p_2q_2 - p_3q_3) + \boldsymbol{k}(p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0).
\end{aligned}$$

3 Representation of Reduced Biquaternion

In what follows we introduce a representation of the reduced biquaternion which have the following form:

$$e_1 = \frac{1}{2}(1 + \boldsymbol{j}) \text{ and } \quad e_2 = \frac{1}{2}(1 - \boldsymbol{j}).$$

Here they satisfy $e_1 + e_2 = 1$ and $e_1 - e_2 = \boldsymbol{j}$. We say that e_1 and e_2 are idempotents because $e_1^2 = e_1$ and $e_2^2 = e_2$. It is not difficult to check that

$$e_1e_2 = 0 \text{ and } e_2e_1 = 0.$$

Below we obtain the important results of the above representation.

Theorem 3.1. *If we write* $q_a = q_0 + q_1\boldsymbol{i}$ *and* $q_b = q_2 + q_3\boldsymbol{i}$, *every reduced biquaternion* $q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3$ *can be represented uniquely by the linear combination of* e_1 *and* e_2, *i.e,*

$$q = q_a + q_b\boldsymbol{j}$$
$$= q_{a+b}e_1 + q_{a-b}e_2,$$

where $q_{a+b} = q_a + q_b$ *and* $q_{a-b} = q_a - q_b$.

Proof. It directly follows from the definition of reduced biquaternion that

$$q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$
$$= (q_0 + q_1\boldsymbol{i}) + (q_2 + q_3\boldsymbol{i})\boldsymbol{j}$$
$$= q_a + q_b\boldsymbol{j}$$
$$= \frac{1}{2}(q_a + q_b + q_a - q_b) + \frac{\boldsymbol{j}}{2}(q_a + q_b - q_a + q_b)$$
$$= \frac{1}{2}(q_a + q_b) + \frac{\boldsymbol{j}}{2}(q_a + q_b) + \frac{1}{2}(q_a - q_b) - \frac{\boldsymbol{j}}{2}(q_a - q_b)$$
$$= (q_a + q_b)\left(\frac{1}{2} + \frac{\boldsymbol{j}}{2}\right) + (q_a - q_b)\left(\frac{1}{2} - \frac{\boldsymbol{j}}{2}\right)$$
$$= q_{a+b}e_1 + q_{a-b}e_2.$$

Theorem 3.2. *If* $q_{a+b} \neq 0$ *and* $q_{a-b} \neq 0$, *then the inverse of the reduced biquaternion* q *have the following form*

$$q^{-1} = (q_{a+b})^{-1}e_1 + (q_{a-b})^{-1}e_2.$$

Proof. Simple computations show that

$$q^{-1} = \frac{1}{q}$$
$$= \frac{1}{q_a + q_b\boldsymbol{j}}$$
$$= \frac{1}{q_a + q_b\boldsymbol{j}}\,\frac{q_a - q_b\boldsymbol{j}}{q_a - q_b\boldsymbol{j}}$$
$$= \frac{q_a - q_b\boldsymbol{j}}{q_a^2 - q_aq_b\boldsymbol{j} + q_aq_b\boldsymbol{j} - q_b^2}$$
$$= \frac{q_a - q_b\boldsymbol{j}}{q_a^2 - q_b^2}.$$

Using the fact that $q_a - q_b\boldsymbol{j} = q_{a+b}e_2 + q_{a-b}e_1$, the above equation can be written as

$$
\begin{aligned}
q^{-1} &= \frac{q_{a+b}e_2 + q_{a-b}e_1}{(q_a + q_b)(q_a - q_b)} \\
&= \frac{q_{a+b}e_2}{q_{a+b}q_{a-b}} + \frac{q_{a-b}e_1}{q_{a+b}q_{a-b}} \\
&= \frac{e_2}{q_{a-b}} + \frac{e_1}{q_{a+b}}.
\end{aligned}
$$

Hence,

$$q^{-1} = (q_{a+b})^{-1}e_1 + (q_{a-b})^{-1}e_2,$$

which proves the theorem.

With the representation of the reduced biquaternion mentioned above, the multiplication can be simplified

$$pq = qp = (p_a + p_b)(q_a + q_b)e_1 + (p_a - p_b)(q_a - q_b)e_2. \tag{4}$$

This can be easily seen from

$$
\begin{aligned}
pq = qp &= (p_{a+b}e_1 + p_{a-b}e_2)(q_{a+b}e_1 + q_{a-b}e_2) \\
&= \big((p_a + p_b)e_1 + (p_a - p_b)e_2\big)\big((q_a + q_b)e_1 + (q_a - q_b)e_2\big) \\
&= \big((p_a + p_b)(q_a + q_b)e_1^2 + (p_a + p_b)(q_a - q_b)e_1e_2\big) \\
&\quad + \big((p_a - p_b) + (q_a - q_b)e_1e_2 + (p_a - p_b)(q_a - q_b)e_2^2\big).
\end{aligned}
$$

Since $e_2e_1 = 0$ we obtain

$$pq = qp = (p_a + p_b)(q_a + q_b)e_1 + (p_a - p_b)(q_a - q_b)e_2.$$

The next, the reduced biquaternion can be expressed by the real representation matrix in the form

$$
\begin{aligned}
q &\cong M_q \\
&= \begin{bmatrix} q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \\ q_2 & -q_3 & q_0 & -q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix}.
\end{aligned} \tag{5}
$$

Now we obtain that the multiplication of the two reduced biquaternions q and p can be rewritten in the form

$$qp = M_q v_p\,,$$

where the vector v_p is given by

$$v_p = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Let us now suppose $M_{m\times n}(\mathbb{R}), M_{m\times n}(\mathbb{C})$ and $M_{m\times n}(RB)$ denote the set of all $m \times n$ matrices over real number $\mathbb{R}$, complex number $\mathbb{C}$ and reduced biquaternionRB, respectively. For $m = n$, the set of real and reduced biquaternion matrices are called square matrices and are denoted by $M_n(\mathbb{R})$ and $M_n(RB)$, respectively. For any $A \in M_{m\times n}(RB)$ may be written as

$$A = A_0 + \boldsymbol{i}A_i + \boldsymbol{j}A_j + \boldsymbol{k}A_k, \tag{6}$$

where A_0, A_i,A_j, and $A_k \in M_{m\times n}(\mathbb{R})$. We therefore can write the above equation in the form

$$A = (A_0 + \boldsymbol{i}A_i) + (A_j + \boldsymbol{i}A_k)\boldsymbol{j}$$

$$= A_1 + \boldsymbol{j}A_2, \tag{7}$$

where $A_1, A_2 \in M_{m\times n}(\mathbb{C})$.

4 Cramer Rule for Reduced Biquaternionic Linear Equation

In this section we shall discuss how to find the solutions of the reduced biquaternionic linear equation of the form

$$Ax = \beta.$$

where $A \in M_n(RB)$, $\beta \in M_{nx1}(RB)$. It is known (see [2]) that for any reduced biquaternion matrix $A = A_0 + \boldsymbol{i}A_i + \boldsymbol{j}A_j + \boldsymbol{k}A_k \in M_n(RB)$ the complex representation is given by

$$\eta(A) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}. \tag{8}$$

The complex matrix $\eta(A) \in \mathbb{C}^{2n\times 2n}$ was called complex representation of A. From equation (8) we can easily obtain adjoint of $\eta(A)$ and is denoted by $\mathrm{Adj}(\eta(A))$.

Using the definition of complex representation we get the solution of $Ax = \beta$ which is corresponding to the solution of $\eta(A)\eta(x) = \eta(\beta)$. It means that $Ax = \beta$ has a solution x if and only if $\eta(A)Y = \eta(\beta)$ has a solution $Y = \eta(x)$. If a reduced biquaternion matrix A is invertible, i,e. $\det(A) = \det(\eta(A)) \neq 0$, then linear equation $\eta(A)Y = \eta(\beta)$ has a unique solution $\eta(x)$ and

$$\eta(x_t) = \frac{1}{\det(A)} \begin{bmatrix} D_{2t-1} & D_{2t} \\ D_{2t} & D_{2t-1} \end{bmatrix}, \qquad t = 1,2, \cdots n, \tag{9}$$

where D_{2t-1} is the determinant obtained by replacing the (2t-1)-th column of $\mathrm{Adj}(\eta(A))$ by the first column of $\eta(\beta)$, dan D_{2t} the determinant obtained by replacing the (2t- 1)-th column by the second column of $\eta(\beta)$. Suppose that

$$\Lambda_t = \left(\eta\left(\begin{bmatrix} D_{2t-1} & D_{2t} \\ D_{2t} & D_{2t-1} \end{bmatrix}\right)\right)^{-1}.$$

Then equation (6) can be written in the form

$x_t = \frac{1}{\det(A)} \Lambda_t, \quad t = 1,2,\dots,n.$

This gives the following important result.

Theorem 4.1 Let $A \in M_n(RB)$, $\beta \in M_{nx1}(RB)$. If A is an invertible reduced biquaternion matrix, then the reduced biquaternionic linear equation $Ax = \beta$ has a unique solution and the solution is given by

$$x_t = \frac{1}{\det(A)} \Lambda_t, \quad t = 1,2,\dots,n.$$

where

$$\Lambda_t = \left(\eta\left(\begin{bmatrix} D_{2t-1} & D_{2t} \\ D_{2t} & D_{2t-1} \end{bmatrix}\right)\right)^{-1}.$$

Especially, if $A \in M_n(\mathbb{C})$ by the definition of complex representation we get $\det(A) = \det(\eta(A)) = \frac{\det A \det A}{\det A}$, and $D_{2t-1} = \Delta_t \det(A)$, $D_{2t} = 0$. This means that we have

$$\begin{aligned} x_t &= \frac{1}{\det(A)} \Lambda_t \\ &= \frac{1}{\det(A)\det(A)} \left(\eta\left(\begin{bmatrix} \Delta_t \det(A) & 0 \\ 0 & \Delta_t \det(A) \end{bmatrix}\right)\right)^{-1} \\ &= \frac{1}{\det(A)\det(A)} \Delta_t \det(A) = \frac{1}{\det(A)} \Delta_t. \end{aligned}$$

Here Δ_t is the determinant obtained by replacing the t-th column of $\det(A)$ by the reduced biquaternion vector β. Therefore, the reduced biquaternionic Cramer rule is a generalization of complex Cramer rule.

Let us illustrate the above result by taking an example of the application of the Cramer rule and Adjoint method. Consider the following reduced biquaternion matrices

$$A = \begin{bmatrix} -i & j \\ 1+k & i+j \end{bmatrix}, \quad \text{and} \quad \beta = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Find all solutions of the reduced biquaternion linear equation $Ax = \beta$.

First, we write the above matrices in the form

$$A = \begin{bmatrix} -i & 0 \\ 1 & i \end{bmatrix} + j \begin{bmatrix} 0 & 1 \\ i & 1 \end{bmatrix}.$$

By the definition of complex representation of the reduced biquaternion matrix we immediately obtain

$$\eta(A)=\begin{bmatrix}-i & 0 & 0 & 1\\ 1 & i & i & 1\\ 0 & 1 & -i & 0\\ i & 1 & 1 & i\end{bmatrix},\quad \eta(\beta)=\begin{bmatrix}1 & 0\\ 0 & 1\\ -i & 0\\ 0 & -i\end{bmatrix}. \tag{10}$$

By direct calculation we get $\det(\eta(A))=-4i\neq 0$. According to (9) we immediately obtain

$$D_1=\begin{vmatrix}1 & 0 & 0 & 1\\ 0 & i & i & 1\\ -i & 1 & -i & 0\\ 0 & 1 & 1 & i\end{vmatrix},\qquad D_2=\begin{vmatrix}-i & 0 & 0 & 1\\ 1 & 1 & i & 1\\ 0 & 0 & -i & 0\\ i & -i & 1 & i\end{vmatrix},\ D_3=\begin{vmatrix}-i & 0 & 1 & 1\\ 1 & i & 0 & 1\\ 0 & 1 & -i & 0\\ i & 1 & 0 & i\end{vmatrix},$$

and

$$D_4=\begin{vmatrix}-i & 0 & 0 & 0\\ 1 & i & i & 1\\ 0 & 1 & -i & 0\\ i & 1 & 1 & -i\end{vmatrix}.$$

Above gives

$$\det(D_1)=2+2i,\ \det(D_2)=-2i-2, \tag{11}$$

and

$$\det(D_3)=-2i+2,\quad \det(D_4)=0. \tag{12}$$

From the reduced biquaternionic Cramer rule, the linear equation has a unique solution, namely,

$$\eta(x_1)=\frac{1}{\det(A)}\begin{bmatrix}D_1 & D_2\\ D_2 & D_1\end{bmatrix}$$

$$x_1=\frac{1}{\det(A)}\left(\eta\left(\begin{bmatrix}D_1 & D_2\\ D_2 & D_1\end{bmatrix}\right)\right)^{-1}. \tag{13}$$

Substitution (11) into (13) yields

$$x_1=\frac{1}{-4i}\left(\eta\left(\begin{bmatrix}2+2i & -2i-2\\ -2i-2 & 2+2i\end{bmatrix}\right)\right)^{-1}$$

$$=\frac{1}{-4i}(2+2i-2k-2j).$$

In a similar way, we have

$$x_2=\frac{1}{\det(A)}\left(\eta\left(\begin{bmatrix}D_3 & D_4\\ D_4 & D_3\end{bmatrix}\right)\right)^{-1}$$

$$x_2=\frac{1}{-4i}\left(\eta\left(\begin{bmatrix}-2i+2 & 0\\ 0 & -2i+2\end{bmatrix}\right)\right)^{-1}$$

$$= \frac{1}{-4i}(-2i + 2).$$

Next, we use the adjoint method to solving the reduced biquaternion linear $Ax = \beta$. From the first term of (10) we easily obtain

$$adj\ (\eta(A)) = \begin{bmatrix} 2+2i & 0 & 0 & -2-2i \\ -1-i & -1-i & -1+i & 1-i \\ 0 & -2-2i & 2+2i & 0 \\ -1+i & 1-i & -1-i & -1-i \end{bmatrix}.$$

Hence,

$$\left(adj\ (\eta(A))\right)^{-1} = \begin{bmatrix} 2+2i & -2j-2k \\ -1-i-j+k & -1-i+j-k \end{bmatrix}.$$

This means that we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-4\mathrm{i}} \begin{bmatrix} 2+2i & -2j-2k \\ -1-i-j+k & -1-i+j-k \end{bmatrix} \begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix}$$

$$= \frac{1}{-4\mathrm{i}} \begin{bmatrix} 2+2i-2j+2k \\ -1-i-j+k++i-1-k-j \end{bmatrix}$$

$$= \frac{1}{-4\mathrm{i}} \begin{bmatrix} 2+2i-2j+2k \\ -2-2j \end{bmatrix}.$$

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